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Generalized stochastic dominance and bad outcome aversion

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Abstract Incomplete preferences over lotteries on a finite set of alternatives satisfying, besides independence and continuity, a property called *bad outcome aversion* are considered. These preferences are characterized in terms of their specific multi-expected utility representations (cf. Dubra et al., J Econ Theory, 115:118–133, 2004), and can be seen as generalized stochastic dominance preferences.

1 Introduction

A familiar and widely accepted way to order probability distributions on a set of alternatives is to use (first or higher degree) stochastic dominance.[1] A typical property of such an incomplete ordering, or preference, is that a positive probability on a bad alternative cannot be compensated by putting high probabilities on better alternatives. In this article, we study and characterize this typical property, which we call *bad outcome aversion* (BOA). Specifically, we consider incomplete preferences over lotteries on a finite set of alternatives and assume the classical conditions of (von

[1] See Levy (1992) for an overview of theory and applications of stochastic dominance.

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Neumann and Morgenstern) independence and continuity, so that the 'multi-expected utility' theorem of Dubra et al. (2004) applies. This result characterizes such preferences in terms of representing closed and convex sets of functions. Our main result (Theorem 3.2) characterizes BOA in terms of specific elements contained in these representing sets of functions. This characterization is very helpful in defining or identifying preferences satisfying BOA.

Thus, we present a broad generalization of stochastic dominance preferences. In Example 3.3, we apply our main result to a class of stochastic dominance preferences derived from Fishburn (1976). An application to stochastic dominance preferences in two-person non-cooperative games can be found in Perea et al. (2006), who extend a result of Fishburn (1978).

This study can also be applied to measuring inequality. Alternatives correspond to states of welfare, for instance, income, and lotteries to distributions of the population over these states. Bad outcome aversion then implies that, for instance, an income distribution ℓ can never be better than another income distribution ℓ' if ℓ has a higher percentage of poorest people than ℓ', not even if under ℓ' all other people are richer than under ℓ.

2 Preliminaries

Let $X := \{x_1, \dots, x_n\}$, where $n \geq 3$, be a finite set of *alternatives* and let $\Delta(X)$ denote the set of probability distributions (lotteries) over X. We also use the letters $x, y, \dots$ to denote elements of X. A *preference* $\succeq$ is a reflexive and transitive binary relation on $\Delta(X)$. If $(p, q) \in \succeq$, then we say that p is (weakly) preferred over q. Instead of $(p, q) \in \succeq$, we often use the notation $p \succeq q$. We write $p \succ q$ if $p \succeq q$ and $q \not\succeq p$, and $p \sim q$ if $p \succeq q$ and $q \succeq p$. For $p \in \Delta(X)$ and $i \in \{1, \dots, n\}$, p_i denotes the probability that p assigns to x_i, and $p(x)$ the probability that p assigns to $x \in X$. The degenerate lottery that assigns probability one to the alternative $x \in X$ is identified with x. Observe that we do not require completeness of $\succeq$.

The following possible conditions on $\succeq$ are well known.

Axiom 2.1 (*Independence*) For all $p, q, r \in \Delta(X)$ and $0 \leq \lambda \leq 1$,

$$p \succeq q \Rightarrow \lambda p + (1-\lambda) r \succeq \lambda q + (1-\lambda) r.$$

Axiom 2.2 (*Continuity*) For all $q \in \Delta(X)$, the sets $\{p \in \Delta(X) | p \succeq q\}$ and $\{p \in \Delta(X) \mid q \succeq p\}$ are closed in $\Delta(X)$.

Let $U \subseteq \mathbb{R}^X$ be a set of real-valued functions on X. For $u \in \mathbb{R}^X$ and a lottery $p \in \Delta(X)$ denote by

$$\mathbb{E}_u(p) := \sum_{i=1}^{n} p_i u(x_i)$$

the expectation of p under u. We say that U *represents* the preference $\succeq$ if for all $p, q \in \Delta(X)$,

$$p \succeq q \Leftrightarrow \mathbb{E}_u(p) \geq \mathbb{E}_u(q) \quad \text{for all } u \in U.$$

The following 'multi-expected utility' theorem follows from Dubra et al. (2004) and generalizes the familiar von Neumann–Morgenstern expected utility theorem to incomplete preferences.[2]

Theorem 2.3 *Let $\succeq$ be a preference. Then $\succeq$ satisfies independence and continuity if and only if there is a closed and convex set $U \subseteq \mathbb{R}^X$ that represents $\succeq$.*

3 Bad outcome aversion

First degree stochastic dominance is a well-known example of a preference to which Theorem 2.3 applies. For any permutation π of $\{1, \ldots, n\}$, the first degree stochastic dominance preference $\succeq_\pi$ is defined by

$$p \succeq_\pi q \Leftrightarrow \sum_{i=1}^{j} p_{\pi(i)} \leq \sum_{i=1}^{j} q_{\pi(i)} \quad \text{for all } j = 1, \ldots, n$$

for all $p, q \in \Delta(X)$. Note that $\succeq_\pi$ strictly orders all alternatives of X, specifically, $x_{\pi(n)} \succ_\pi \ldots \succ_\pi x_{\pi(1)}$. Therefore, first degree stochastic dominance preferences are complete on degenerate lotteries. Clearly, if $q_{\pi(1)} < p_{\pi(1)}$, then $p \not\succeq q$: thus if p puts higher probability on the worst alternative than q, then this can never be compensated by p putting higher probabilities on better alternatives. This property is typical for a preference such as first degree stochastic dominance. In a more general form, it is described by the following axiom.

Axiom 3.1 (*Bad outcome aversion, BOA*) For all $p, q \in \Delta(X)$ and all $x \in X$, if $p(x) > q(x)$ and $p(z) = q(z)$ for all $z \in X$ with $x \succeq z$, then $p \not\succeq q$.

The interpretation of this axiom is as follows. Think of x as a 'bad' alternative, on which p puts more weight than q and such that p and q put equal weights on all alternatives worse than x. Then, the axiom says that this can never be compensated – that is, p cannot be made to dominate q – by the weights put by p on alternatives that are better or at least not worse than x.

In the framework of inequality measurement, the alternatives are social states, and the probabilities are population shares. Bad outcome aversion then implies that if in p the number of people in state x is higher than in q while these numbers are equal for all worse states, then p can only be worse than q, never better. In other words, if there are more people in a poor state, then this fact cannot be compensated by having more people in some richer state. Thus, BOA has a typically Rawlsian flavor.

[2] Dubra et al. (2004) formalize observations already present in Aumann (1962).

Clearly, first degree stochastic dominance preferences satisfy BOA, but there are many more. The purpose of this note is to characterize BOA for preferences that satisfy independence and continuity and that strictly order all elements of X, by using the multi-expected utility theorem, Theorem 2.3. More precisely, we show that such a preference satisfies BOA if and only if the representing class of functions contains specific elements.[3]

Theorem 3.2 *Let the preference $\succeq$ satisfy independence and continuity, and suppose $x_n \succ x_{n-1} \succ \ldots \succ x_1$. Let $U \subseteq \mathbb{R}^X$ represent $\succeq$. Then, $\succeq$ satisfies BOA if and only if for each $i = 1, \ldots, n-1$ there is a sequence $(u^k)_{k\in\mathbb{N}}$, $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$, such that*

$$\lim_{k\to\infty} \frac{u^k(x_n) - u^k(x_{i+1})}{u^k(x_n) - u^k(x_i)} = 0. \tag{1}$$

The intuition for condition (1) is as follows. For every alternative x_i that is not the best alternative x_n, we can make the ratio of the utility difference between x_n and x_i to the utility difference between x_n and the (one-step) better alternative x_{i+1}, as large as we want. This implies that this difference can never be 'compensated' by putting high probability on x_{i+1} (or better alternatives) whenever x_i receives positive probability. This is, indeed, what BOA is intended to capture.

Proof of Theorem 3.2 We may normalize any $u \in U$ such that $u(x_1) = 0$ and $u(x_n) = 1$.

For the 'if' part, let p, q, x satisfy the conditions in the statement of BOA. Then, $x \neq x_n$, so $x = x_i$ for some $i \in \{1, \ldots, n-1\}$. Let $(u^k)_{k\in\mathbb{N}}$ be a sequence with $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$ such that (1) is satisfied. Without loss of generality, we may assume that the sequence $(u^k)_{k\in\mathbb{N}}$ converges. With $\alpha := \sum_{j=1}^{i-1} q_j = \sum_{j=1}^{i-1} p_j$, we can write

$$\lim_{k\to\infty} \mathbb{E}_{u^k}(q) \geq \lim_{k\to\infty} \sum_{j=1}^{i-1} q_j u^k(x_j) + q_i u^k(x_i) + (1 - q_i - \alpha) u^k(x_{i+1})$$

and

$$\lim_{k\to\infty} \mathbb{E}_{u^k}(p) \leq \lim_{k\to\infty} \sum_{j=1}^{i-1} p_j u^k(x_j) + p_i u^k(x_i) + (1 - p_i - \alpha).$$

We claim that for k sufficiently large, $\mathbb{E}_{u^k}(q) > \mathbb{E}_{u^k}(p)$. In order to show this, since $p_j = q_j$ for $1 \leq j \leq i-1$, it is sufficient to prove that

$$q_i u^k(x_i) + (1 - q_i - \alpha) u^k(x_{i+1}) > p_i u^k(x_i) + (1 - p_i - \alpha)$$

[3] Typically also, lexicographic preferences on $\Delta(X)$ satisfy BOA, but they are not continuous. It is not hard to show that any preference which satisfies BOA and allows no indifference between the alternatives in X, is a subset of a lexicographic preference.

or, equivalently

$$(1 - q_i - \alpha)[1 - u^k(x_{i+1})] < (p_i - q_i)[1 - u^k(x_i)]$$

for k sufficiently large. This, however, follows by (1). So $p \not\succeq q$.

For the converse, assume that $\succeq$ satisfies BOA. Fix $i \in \{1, \ldots, n-1\}$. Fix $0 < \pi < 1$, consider the lottery $p = \pi x_i + (1-\pi)x_{i+1}$, and for each $0 < \varepsilon < 1 - \pi$ consider the lottery $p^\varepsilon = (\pi + \varepsilon)x_i + (1 - \pi - \varepsilon)x_n$. By BOA, $p^\varepsilon \not\succeq p$, and, hence, there is a $u^\varepsilon \in U$ such that $\mathbb{E}_{u^\varepsilon}(p) > \mathbb{E}_{u^\varepsilon}(p^\varepsilon)$, i.e.,

$$(1-\pi)(1 - u^\varepsilon(x_{i+1})) < \varepsilon(1 - u^\varepsilon(x_i)).$$

This implies the existence of a sequence $(u^k)_{k\in\mathbb{N}}$ with $u^k \in U$ and $u^k(x_i) < u^k(x_n)$ for each $k \in \mathbb{N}$ such that (1) is satisfied. □

A consequence of Theorem 3.2 is that, under the additional conditions in the theorem, BOA implies incompleteness of the preference. This is so because a complete preference is represented by a unique 0–1 normalized function and, thus, a sequence satisfying (1) cannot exist for every i.

In the following application of Theorem 3.2, we consider a special class of preferences.

Example 3.3 For each $t \in \mathbb{R}, t \geq 1$, we define the $n \times n$-matrix A^t by $\left(a^t\right)_{ij} := 0$ for all i, j with $i > j$ and by $\left(a^t\right)_{ij} := -\Gamma\left(t + j - i\right) / \left(j - i\right)!\Gamma(t)$ for all i, j with $i \leq j$.[4] We define the preference $\succeq_t$ by

$$p \succeq_t q \Leftrightarrow pA^t \geq qA^t$$

for all $p, q \in \Delta(X)$, where the inequality is coordinate-wise. It can be verified that $x_n \succ_t \ldots \succ_t x_1$, and thus $\succeq_t$ strictly orders the elements of X. Also, $\succeq_1$ is the first degree stochastic dominance preference associated with this ordering of the elements of X, i.e., $\succeq_1$ is equal to $\succeq_\pi$ for π equal to the identity. The preferences $\succeq_t$ were introduced in Fishburn (1976) as a generalization of first and higher degree stochastic dominance with the real line as set of alternatives. Here, we have adapted Fishburn's definition to our framework of finitely many alternatives. One can think of the alternatives as located on the real line with x_i placed at point i. For instance, the preference $\succeq_2$ is then second-degree stochastic dominance. In general, the representing set U as in Theorem 2.3 for $\succeq_t$ is the convex hull of the columns of A^t. In this case, one can simply take u^k in Theorem 3.2 constant and equal to the i-th column of A^t.

We conclude with an examination of the three alternative case.

Example 3.4 For the case of three alternatives the consequences of Theorem 3.2 are as follows. Since we can assume that every $u \in U$ has the form $(0, \alpha, 1)$, the theorem applied for $i = 1$ implies $(0, 1, 1) \in U$. Let $\alpha^* := \inf\{\alpha \mid (0, \alpha, 1) \in U\}$, then

[4] Here, $\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds$ is the gamma function. In particular, $\Gamma(t) = (t-1)!$ for $t \in \mathbb{N}$.

convexity and closedness of U imply that U is the convex hull of $(0, 1, 1)$ and $(0, \alpha^*, 1)$. Consider, on the other hand, the 3×3-matrix A^t and normalize its columns. This results in the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{t-1}{t} & \frac{t}{t+2} \\ 1 & 1 & 1 \end{pmatrix},$$

implying that the class of functions U^t representing $\succeq_t$ is the convex hull of $(0, 1, 1)$ and $(0, (t-1)/t, 1)$ if $t \le 2$ and of $(0, 1, 1)$ and $(0, t/(t+2), 1)$ if $t \ge 2$. In turn, this implies that $\succeq$ coincides with $\succeq_t$, where $t = 1/(1-\alpha^*)$ if $\alpha^* \le 1/2$ and $t = 2\alpha^*/(1-\alpha^*)$ if $\alpha^* \ge 1/2$. Thus, for $n = 3$, if $\succeq$ satisfies independence, continuity, BOA, and if $x_3 \succ x_2 \succ x_1$, then $\succeq = \succeq_t$ for some $t \in \mathbb{R}$, $t \ge 1$.

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